

Euler configurations and quasi-polynomial systems

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Abstract. In the Newtonian 3-body problem, for any choice of the three masses, there are exactly three Euler configurations (also known as the three Euler points). In Helmholtz' problem of 3 point vortices in the plane, there are at most three collinear relative equilibria. The “at most three” part is common to both statements, but the respective arguments for it are usually so different that one could think of a casual coincidence. By proving a statement on a quasi-polynomial system, we show that the “at most three” holds in a general context which includes both cases. We indicate some hard conjectures about the configurations of relative equilibrium and suggest they could be attacked within the quasi-polynomial framework.

1. Introduction

Statics is the science that studies the equilibria of a mechanical system. The notion of *relative equilibrium* generalizes the notion of equilibrium in the case of a mechanical system with a continuous symmetry group. This group is often a group of rotations. In a motion of relative equilibrium the configuration is not required to be fixed, as for an equilibrium. But it is fixed “up to symmetry”, i.e. often “up to rotation”.

The two examples we start with are basic and well-known. In 1762, Euler [Eu1] considered the Sun, the Earth and the Moon as three point particles moving under Newtonian gravitation. He discovered a type of motion where the three bodies permanently form a collinear configuration. Each particle describes an elliptic trajectory. The trajectories may also be hyperbolic, parabolic, circular or rectilinear. In the circular case, we have a relative equilibrium of the 3-body problem. In one of the possibilities described by Euler, the Moon is four times farther from the Earth than it is today, and from the Earth it is seen as a permanent full moon. Many philosophical debates about the perfection of the world were raised by Euler's permanent full moon. They were closed by Liouville who proved that the relative equilibrium is unstable [Lut].

The second example is the problem of three Helmholtz' vortices. We consider a perfect fluid with an infinite horizontal surface and a constant thickness. Some states of the fluid are completely described giving the positions and the vorticities of a finite number of "point vortices". These vortices are centered on a vertical line and are sometimes called line vortices. We forget the vertical direction and just describe the vortices as points in the horizontal plane. Helmholtz found the ordinary differential system modeling the motion of the vortices. There exist relative equilibria, i.e. configurations that remain unchanged up to rotation and translation. This is possible with a collinear configuration of N vortices.

Some famous experiments by A.M. Mayer simulate Helmholtz equations by a device where N identical magnets are floating on a surface of water. This reminds us that equations similar to Helmholtz' are quite frequent in physical models. Mayer found several equilibrium configurations. He did not detect any of the collinear equilibria, which are unstable for any $N \geq 3$, as are the collinear equilibria of N equal vortices (see [Are] and [ANS]).

In each example we want to determine the set of relative equilibria. To do so we have to solve a system of equations. Here as in other situations the system may be reduced to a polynomial system, and we are interested in the real solutions.

Real solutions of equations. There are many histories of the theory of equations. The most popular ones relate how blind were our ancestors who in the 15th century ignored the negative roots of a polynomial equation. They continue telling the discovery of complex numbers, the fundamental theorem of algebra, and the Galois group of a polynomial equation with rational coefficients. This is actually the history of how the initial questions were forgotten.

The initial questions were about the real solutions, often the positive ones. This is true for the question of determining the equilibria of a mechanical system. It is also true for the geometrical, optical or accountancy problems which since the antiquity have motivated the study of complicated equations.

If *some* traditions did forget the real questions, others studied them carefully. A central achievement is Descartes' rule of signs. It bounds the number of positive solutions of a polynomial equation $\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0$ by the number of sign variations in the list $\alpha_0, \alpha_1, \dots, \alpha_n$ of its real coefficients. In his famous works on equations, Lagrange introduced the reasoning on the permutation of roots which lead to Galois theory. But he also developed completely distinct ideas, presenting methods to get the exact number of real solutions if the coefficients are given numbers. In [Lagr], he emphasizes that he does not know how to discuss the number of real solutions if some coefficients of the equation depend on parameters.

Important advances are due to 19th century mathematicians. Fourier noticed that if we replace Descartes' list of coefficients with the "list of successive derivatives at a point", we get an upper bound for the number of roots in any given interval. Vincent showed that a clever use of rational transformations allows to decrease the number of sign variations. Sturm changed Fourier's list into the "list of values at a point of the successive remainders in the Euclidean algo-

rithm applied to the polynomial and its derivative”, and got the exact number of roots in an interval. Hermite found a quadratic form the signature of which is the number of real roots of the given equation. However, despite many recent continuations of these classical works, the discussion of the number of real roots when some parameters vary remains difficult.

Laguerre [Lagu] developed Descartes’ rule in another direction. We consider the expression

$$P(x) = \alpha_1 x^{\beta_1} + \cdots + \alpha_n x^{\beta_n}, \quad \text{with } (\alpha_1, \dots, \beta_n) \in \mathbb{R}^{2n}. \quad (1.1)$$

Laguerre proved that the number of positive roots of the equation $P(x) = 0$ is not greater than the number of sign variations in the list $\alpha_1, \dots, \alpha_n$, assuming that the monomial terms of P are ordered in such a way that $\beta_1 < \cdots < \beta_n$. Of course we ignore the null α_i ’s in the count of sign variations. The proof is: let i be the first index such that the sign of α_i is not the sign of α_1 . Consider $Q(x) = (x^{-\beta_i} P(x))'$. Count the sign variations in the list $\alpha_1(\beta_1 - \beta_i), \dots, \alpha_n(\beta_n - \beta_i)$ of the coefficients of Q . Compare to P : there is one sign variation less. Use a recurrence hypothesis on the number of sign variations, apply Rolle’s theorem and conclude.

Laguerre’s statement includes Descartes’ rule of signs and is just as simple. It leads us out of the world of algebraic equations. We accept irrational exponents, even if this seems useless in the applications. The interesting new feature is that the exponents may be varied continuously. The natural observation that the number of real roots remains bounded while the exponents are varied is now understandable.

Our relative equilibria are given by an equation where an exponent called b varies. When $b = -1$ the equation is algebraic of degree 3 and defines relative equilibria of vortices. When $b = -2$ it has degree 5 and defines central configurations of celestial bodies. Simple and sharp statements about the number of real solutions in these problems belong to the theory of Laguerre’s type systems. This is also the theory of quasi-polynomial systems, as we will explain in §6.

2. Euler Configurations

2.1. *The equations of motion.* Let x_i and m_i be respectively the abscissa and the mass of the particle i , and call $(x_1, \dots, x_n) \in \mathbb{R}^n$ a *collinear configuration*. We assume $(m_1, \dots, m_n, b) \in \mathbb{R}^{n+1}$ and set

$$\gamma_i = \sum_{k \neq i} m_k \rho(x_{ki}), \quad x_{ki} = x_i - x_k, \quad \rho(x) = |x|^{b-1}. \quad (2.1)$$

If $b = -2$, $x_{ij} \neq 0$ for any i, j , $1 \leq i < j \leq n$, and $m_i > 0$ for any i , the Newtonian equations for the particles on the line are

$$\ddot{x}_i = -\gamma_i. \quad (2.2)$$

There is a second physical interpretation: we consider the n collinear particles as Helmholtz’ vortices in the Euclidean plane, with vorticities $m_i \in \mathbb{R}$, and

ordinates $y_i = 0$. Then Helmholtz' law is $\dot{x}_i = 0$, $\dot{y}_i = \gamma_i$ where γ_i is given by Formula (2.1) with $b = -1$. We assume $x_{ij} \neq 0$ for any i, j , $1 \leq i < j \leq n$.

2.2. Relative equilibria. Central configurations. Euler's and Moulton's configurations. The collinear central configurations are, by definition, the collinear configurations (x_1, \dots, x_n) such that there exists a $\lambda \in \mathbb{R}$ with

$$\gamma_j - \gamma_i = \lambda x_{ij} \quad 1 \leq i < j \leq n.$$

They are also called *Moulton configurations*, and in the case $n = 3$, *Euler configurations*. These terminologies come from Celestial Mechanics.

In the Newtonian n -body problem, if we associate to an initial Moulton configuration (x_1, \dots, x_n) the initial velocities $(\dot{x}_1, \dots, \dot{x}_n) = \nu(x_1, \dots, x_n)$, where $\nu \in \mathbb{R}$, the motion is homothetic. If we associate to the configuration in a plane $((x_1, 0), \dots, (x_n, 0))$ the velocities $\sqrt{\lambda}((0, x_1), \dots, (0, x_n))$, the motion is of relative equilibrium. In the Helmholtz problem, where $b = -1$, a collinear central configuration of vortices has a motion of relative equilibrium: during the motion, the distances between particles remain constant.

The conditions for central configuration express that the n -uple (x_1, \dots, x_n) and the n -uple $(\gamma_1, \dots, \gamma_n)$ are equal up to a translation and a scaling.

This may be written as

$$P_{ijk}(x_1, \dots, x_n) = \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ \gamma_i & \gamma_j & \gamma_k \end{vmatrix} = 0, \quad 1 \leq i < j < k \leq n. \quad (2.3)$$

Translating or rescaling a configuration solution of (2.3) we obtain other solutions. If $n = 3$ we normalize the configurations by putting $x_1 = 0$, $x_2 = 1$, $x_3 = 1 + s$. The equation for normalized Euler's configurations is the single equation (2.3), written as

$$g(s) = P_{123}(0, 1, 1 + s) = 0. \quad (2.4)$$

Euler stated and proved the existence and uniqueness of a central configuration of three particles of given positive masses and given ordering. Here is the statement in our notation.

2.3. Proposition. If $m_i > 0$, $i = 1, 2, 3$, Equation (2.4) with $b = -2$ has one and only one positive solution.

The proof appears in §8 of [Eu2]. Assuming $s > 0$, we get

$$\begin{aligned} (1+s)^2 s^2 g(s) &= m_1 s^2 (1 - (1+s)^3) + m_2 (1+s)^2 (1 - s^3) + m_3 ((1+s)^3 - s^3) \\ &= -(m_1 + m_2) s^5 - (3m_1 + 2m_2) s^4 - (3m_1 + m_2) s^3 + (m_2 + 3m_3) s^2 + (2m_2 + 3m_3) s + m_2 + m_3. \end{aligned} \quad (2.5)$$

Euler writes "eumque unicum elici, cum unica signorum variatio occurat". The sequence of coefficients changes sign exactly once. By Descartes' rule of signs there is exactly one positive root. The success of this argument is surprising: it is not often that Descartes' rule gives an answer for all the required values of

the parameters. Moreover, we check that the argument works for any negative integer value of b . In particular we put $b = -1$ and find

$$\begin{aligned} (1+s)sg(s) &= m_1 s(1-(1+s)^2) + m_2(1+s)(1-s^2) + m_3((1+s)^2 - s^2) \\ &= -(m_1+m_2)s^3 - (2m_1+m_2)s^2 + (m_2+2m_3)s + m_2+m_3. \end{aligned} \quad (2.6)$$

We notice that for the odd integer values of b the function ρ in (2.1) is simplified as $\rho(x) = x^b$. Contrary to (2.5), the rational expression (2.6) of $g(s)$ is valid for $s < 0$. We get this other result.

2.4. Proposition. For any given $(m_1, m_2, m_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$, (2.4) with $b = -1$ has at most three roots on its domain of definition $\mathbb{R} \setminus \{-1, 0\}$.

In less precise words, there are at most three central configurations of three collinear vortices, for given vorticities but free ordering of the vortices (in our favorite language given masses but free ordering of the particles). Note that we are always thinking of “distinguishable particles”. Counting configurations of indistinguishable particles gives more complicated results (see [LoS]).

In both cases $b = -1$ and -2 , Euler’s argument gives *exactly* three central configurations, one for each ordering, if $m_i > 0$, $i = 1, 2, 3$. If the signs of the m_i are arbitrary, we use the argument of the degree. For $b = -1$ this gives Proposition 2.4. For $b = -2$ it gives 15 as the upper bound for the number of central configurations: there is a different polynomial (2.5) of degree 5 for each of the 3 orderings. These crude arguments give the upper bounds:

b	-1	-2	-3	-4	
bound $m_i \in \mathbb{R}$	3	15	7	27	(2.7)
bound $m_i > 0$	3	3	3	3	

The first bound is from the most elementary *algebraic geometry*. It uses the upper bound on the number of roots given by the degree. The second bound is from the most elementary *real algebraic geometry*. It uses Descartes’ rule of signs. The word algebraic indicates that we consider polynomial equations, and treat them staying in the framework of polynomial techniques.

The bound 15 in Table (2.7) is far from optimal. We ask the question:

2.5. Question. Fix $b = -2$ and consider all the possible values of the parameter $(m_1, m_2, m_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. What is the maximal number of roots of Equation (2.4) in the domain $\mathbb{R} \setminus \{-1, 0\}$?

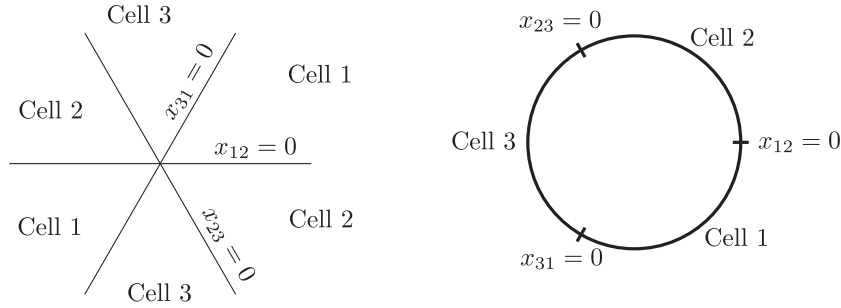
To answer this question, we found better to forget the polynomial expansion (2.5) and to prove that 3 is the maximum number of roots for any *real* negative b . We use the most elementary *real analysis*, based on Rolle’s theorem. Paradoxically this simplifies the quest for a proof. For non-integer exponents there are less available techniques, and the successful ones are among them. We need less attempts to determine which is the technique that works.

3. Generalized Euler’s configurations. The results.

We describe the set of Euler configurations, i.e. collinear central configurations of three particles, under the general hypothesis $(m_1, m_2, m_3, b) \in \mathbb{R}^4$.

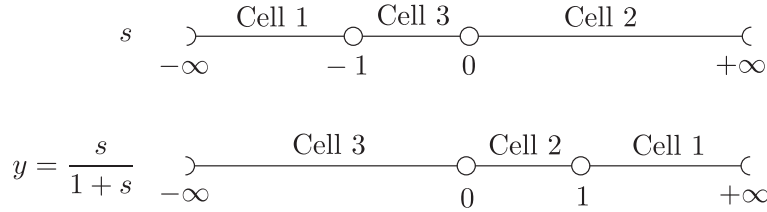
3.1. *The three cells.* There are 6 possible orderings of the particles, and the number becomes 3 after identifying an ordering with its reversed ordering. We say that there are 3 *cells*, one for each pair of orderings, in the space of collinear configurations. Each cell, i.e. each pair of orderings, has the name of the particle in the middle.

For example the second cell corresponds to the orderings $x_1 < x_2 < x_3$ and $x_3 < x_2 < x_1$. We always consider strict orderings. *By convention the configurations with collision are not central configurations.*



3.2. *Figure*

We already used the parameterization $(x_1, x_2, x_3) = (0, 1, 1 + s)$. Later we will also use the parameterization $(x_1, x_2, x_3) = (0, 1 - y, 1)$. In these parameterizations the cells are indicated below.



3.3. *Figure*

3.4. *Definition.* Let g be the function of s defined at (2.4). We denote by \mathcal{E}_i the number of zeros of g which are strictly inside Cell i . We put $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$.

Thus \mathcal{E}_i is the number of “classes” of Euler configurations with particle i in the middle (two configurations are in the same class if they are homothetic, with a positive or a negative factor.) We present bounds on \mathcal{E}_i and then on \mathcal{E} . If there is no restriction on the masses, the situation is the same for all three cells. We only describe the second cell.

3.5. *Proposition.* Let m_1, m_2, m_3 be arbitrary real masses. The function g vanishes identically on $]0, +\infty[$ only in the following cases: (i) $m_1 = m_2 = m_3 = 0$, (ii) $b = 0$ and $m_1 = -m_2 = m_3$, (iii) $b = 1$, (iv) $b = 2$, $m_2 = 0$ and $m_1 = m_3$, (v) $b = 3$ and $m_1 = m_2 = m_3$.

3.6. *Theorem.* For any $(m_1, m_2, m_3, b) \in \mathbb{R}^4$, we have $\mathcal{E}_2 \leq 3$, except for the

(m_1, m_2, m_3, b) characterized in Proposition 3.5 for which $\mathcal{E}_2 = \infty$.

We continue discussing the value of $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$. Our main applications being $b = -2$ and $b = -1$, we decided to restrict the study to the case $b < 1$.

3.7. Lemma. Suppose $(m_1, m_2, m_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and $b < 1$. If $m_1 m_3 \leq 0$, $\mathcal{E}_2 \leq 1$. If $m_1 \geq 0$, $m_2 \geq 0$, $m_3 \geq 0$, $m_1 + m_2 > 0$, $m_2 + m_3 > 0$ then $\mathcal{E}_2 = 1$. If $\mathcal{E}_2 \geq 2$, there is no zero mass, the exterior masses m_1 and m_3 have the same sign, and the central mass m_2 has the opposite sign. If $\mathcal{E}_2 \geq 2$ and $b < 0$ then moreover $\inf(|m_1|, |m_3|) < |m_2|$.

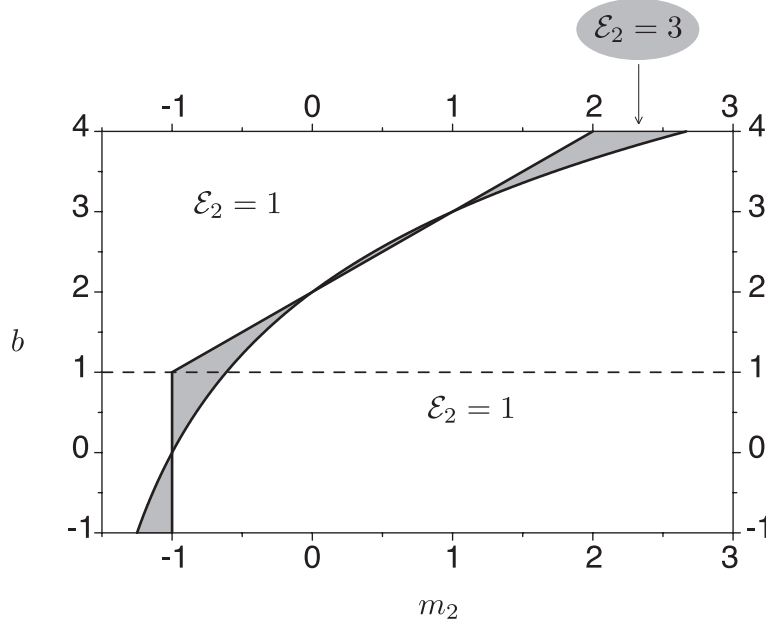
3.8. Theorem. Suppose $(m_1, m_2, m_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. If $0 < b < 1$, $\mathcal{E} \leq 5$. If $b < 0$, $\mathcal{E} \leq 3$.

3.9. Theorem. If $b < 1$, $m_1 > 0$, $m_2 > 0$, $m_3 \geq 0$, $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = 1$, $\mathcal{E} = 3$.

3.10. Proposition. Suppose $(m_1, m_2, m_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ and $b = 0$. If $m_1 = -m_2 = m_3$, then $\mathcal{E}_1 = \mathcal{E}_3 = 0$, $\mathcal{E}_2 = \infty$. If we are not in this case, nor in the similar cases $m_1 = m_2 = -m_3$, $m_1 = -m_2 = -m_3$, $\mathcal{E}_1 \leq 1$, $\mathcal{E}_2 \leq 1$, $\mathcal{E}_3 \leq 1$.

We present a complete discussion of the \mathcal{E}_i 's and of the \mathcal{E} in the particular case where two masses are equal. All the real values of b are considered.

3.11. Proposition. If $m_1 = m_3 = 1$, the number \mathcal{E}_2 is given in Figure 3.12. The frontiers between the different regions are two half-lines starting from $(m_2, b) = (-1, 1)$ and the curve $m_2 = (2^b - 2b)/(b - 1)$. On the frontiers, excluding the three intersections of the curve with the half-lines, $\mathcal{E}_2 = 1$.



3.12. Figure. Case $m_1 = m_3 = 1$.

Remark. By Proposition 3.5, $\mathcal{E}_2 = \infty$ at the excluded points, as well as on the

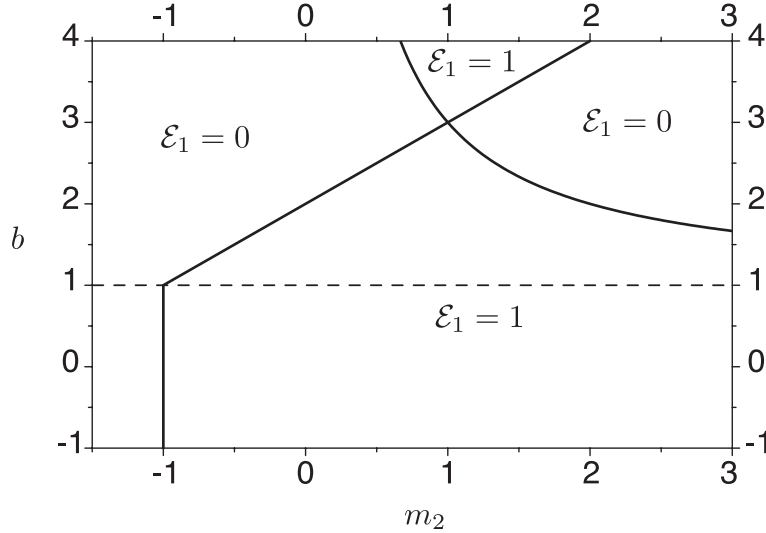
line $b = 1$. Actually $b = 1$ is no longer an exception if we consider $G = g/(b-1)$ instead of g . For $b = 1$,

$$G(s) = m_1(1+s)\log(1+s) + m_2s\log(s) + m_3(1+s)s(\log(s) - \log(1+s)).$$

If we study G , $b = 1$ is not a frontier. It is a frontier if we study g . We symbolized this by a dashed line in Figures 3.12 and 3.14.

3.13. *Proposition.* If $m_1 = m_3 = 1$, the number \mathcal{E}_1 is given in Figure 3.14. The frontiers between the different regions are two half-lines, the same as in Proposition 3.11, and the upper branch of the hyperbola $m_2(b-1) = 2$. On the frontiers, excluding the intersection of the upper half-line with the branch of hyperbola, $\mathcal{E}_1 = 0$.

In the case $m_1 = m_3$ we have $\mathcal{E}_1 = \mathcal{E}_3$. The total number $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$ of Euler configurations is obtained by a mere superposition of Figures 3.12 and 3.14. It gives Figure 3.17. On a frontier \mathcal{E} is the minimum of the \mathcal{E} 's of both regions the frontier separates, if these \mathcal{E} 's are distinct. There are also frontiers for which $\mathcal{E} = 3$ on both sides. On these frontiers $\mathcal{E} = 1$.

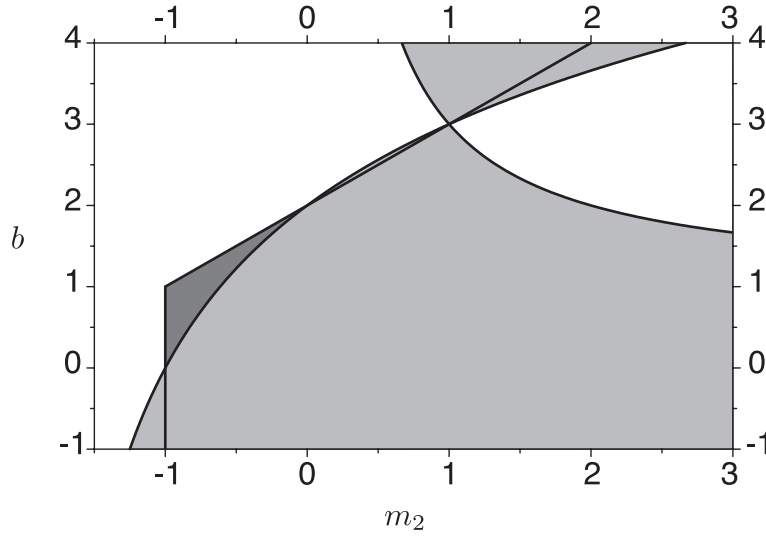


3.14. *Figure.* Again $m_1 = m_3 = 1$.

3.15. *About lower bounds.* We did not find any lower bound of \mathcal{E}_2 or \mathcal{E} . Figure 3.17 suggests $\mathcal{E} \geq 1$. Actually 1 is a “generic” lower bound for \mathcal{E} . If we think of g as a function defined on the circle of Figure 3.2 (the projective line), then its sign changes even number of times. Since, generically speaking, each of the three collisions between particles gives a sign change, \mathcal{E} is an odd number. However, a collision may be or behave like a double root, and so, an even \mathcal{E} , 0 in particular, is not impossible. For example, if $b \neq 1$ and $(m_1, m_2, m_3) = (0, -1, 1)$, there is no Euler configuration in any cell. This may be deduced easily from the expressions of $A - B$ and $B - C$ we will write in 5.5. Let us mention an interesting related result (see [Cel]). For any Euler configuration with $m_1 + m_2 + m_3 = 0$, we have

$m_1x_1+m_2x_2+m_3x_3=0$. If some mass is non-zero, this condition fixes the shape of the configuration, which is non-collisional if and only if $m_1m_2m_3 \neq 0$. As a consequence, for masses satisfying $m_1+m_2+m_3=0$, $\mathcal{E}=1$ if $m_1m_2m_3 \neq 0$, $\mathcal{E}=0$ if $m_1m_2m_3=0$.

3.16. *About the cases with $\mathcal{E} = +\infty$.* As Euler noticed in [Eu2] and [Eu3], the equation of motion (2.2) may be integrated if the initial configuration is an Euler configuration, and if the initial velocities are velocities in some homothetic motion. In the cases where all the configurations are central, listed in Proposition 3.5, the problem is integrable, for any initial velocity. This can be easily checked in each case. One can also notice that if the left hand side of Figure 3.2 is interpreted as a “plane of motion”, then Equation (2.2) in the cases 3.5 defines a central force problem, and the constant of areas is a first integral. Case (iii) is a harmonic oscillator. It is integrated in Proposition 64 of Newton’s *Principia*. Case (v) is discussed by Yoshida [Yos], who looked for all the integrable cases of the problem of three particles with equal masses on a line, moving according to (2.2).



3.17. *Figure.* The three orderings together with $m_1 = m_3 = 1$. White: $\mathcal{E} = 1$. Grey: $\mathcal{E} = 3$. Dark grey: $\mathcal{E} = 5$ (the thinnest region is grey).

4. The formulas

System (2.3) defines the collinear central configurations. We compute the term in m_i , the notation being again $x_{ij} = x_j - x_i$ and $\rho(x) = x|x|^{b-1}$,

$$\begin{aligned} m_i \begin{vmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ 0 & \rho(x_{ij}) & \rho(x_{ik}) \end{vmatrix} &= m_i (x_{ij}\rho(x_{ik}) - x_{ik}\rho(x_{ij})) \\ &= m_i x_{ij} x_{ik} (|x_{ik}|^{b-1} - |x_{ij}|^{b-1}). \end{aligned}$$

In the case $n = 3$, the system is the single equation

$$m_1 x_{31} x_{12} (|x_{12}|^{b-1} - |x_{31}|^{b-1}) + m_2 x_{12} x_{23} (|x_{23}|^{b-1} - |x_{12}|^{b-1}) + m_3 x_{23} x_{31} (|x_{31}|^{b-1} - |x_{23}|^{b-1}) = 0.$$

We normalize the configuration putting $x_1 = 0$, $x_2 = 1$, $x_3 = 1 + s$ and arrive at the previous equation (2.4). We make

$$A = (1+s)(|1+s|^{b-1} - 1), \quad B = s(|s|^{b-1} - 1), \quad C = s(1+s)(|s|^{b-1} - |1+s|^{b-1}).$$

Equation (2.4) is $g(s) = 0$ with

$$g(s) = m_1 A + m_2 B + m_3 C. \quad (4.1)$$

We restrict to the second cell $s > 0$ (see Figure 3.3). The following expansion of g is easy to differentiate several times:

$$g(s) = (m_2 + m_3)s^b + (m_1 + m_3)(1+s)^b + m_3(s^{b+1} - (1+s)^{b+1}) - m_1(1+s) - m_2 s. \quad (4.2)$$

A wonderful tool for the study of the equation $g(s) = 0$ is the second derivative $g''(s)$. We get

$$\frac{g''(s)}{b(b-1)} = (m_2 + m_3)s^{b-2} + (m_1 + m_3)(1+s)^{b-2} + m_3 \frac{b+1}{b-1} (s^{b-1} - (1+s)^{b-1}). \quad (4.3)$$

As we recalled in the introduction, Laguerre studied the functions of $y > 0$ defined as sums of generalized monomials αy^β , where α and β are real numbers. There exists a rational transformation that gives this form to (4.3). We write:

$$g''(s) = (1-y)^{1-b} H(y), \quad \text{with} \quad s = \frac{y}{1-y}, \quad \text{and} \quad H = b(b-1)h, \quad (4.4)$$

where

$$h(y) = -\left(m_2 - \frac{2m_3}{b-1}\right)y^{b-1} + (m_2 + m_3)y^{b-2} - (m_1 + m_3)y + \left(m_1 - \frac{2m_3}{b-1}\right). \quad (4.5)$$

We have $h(1) = 0$. The value $y = 1$ corresponds to $s = \infty$ which is the collision $x_{12} = 0$. The correspondence between y and s is illustrated by Figure 3.3.

5. The proofs

Proof of Proposition 3.5. We first consider the cases $b = 0, 1, 2, 3$. If $b = 0$, $s > 0$, $g(s) = m_2 + m_3 - s(m_1 + m_2)$, which is identically zero if and only if $m_1 = -m_2 = m_3$. If $b = 1$, $g(s) \equiv 0$. If $b = 2$, $g(s) = (m_1 - m_2 - m_3)s + (m_1 + m_2 - m_3)s^2$, identically zero if and only if $m_1 = m_3$ and $m_2 = 0$. If $b = 3$, $g(s) = (2m_1 - m_2 - m_3)s + 3(m_1 - m_3)s^2 + (m_1 + m_2 - 2m_3)s^3$, identically zero if and only if $m_1 = m_2 = m_3$. These are the cases (ii) to (v). For $b \in \mathbb{R} \setminus \{0, 1, 2, 3\}$, $g \equiv 0$ implies $b^{-1}(b-1)^{-1}g'' \equiv 0$, i.e. $h \equiv 0$ for $0 < y < 1$,

and the four monomials in (4.5) have distinct exponents. The four coefficients in (4.5) must be zero. This happens only in two cases. One is $m_1 = m_2 = m_3 = 0$ giving the trivial case (i) and the other $b = -1, m_1 = m_2 = -m_3 \neq 0$ implying $g(s) = m_3(1 + 2s) \neq 0$. QED.

We present two useful lemmas. They are variations of the various statements one proves starting from Rolle's theorem. As we use repeatedly statements as "the function $y \mapsto \alpha y + \beta$ has at most one root" and do not want to repeat each time "except if $(\alpha, \beta) = (0, 0)$ ", we introduce a special way to count the roots.

5.1. Definition. Let $] \mu, \nu[\subset \mathbb{R}$ be an open interval (bounded or not). If $h :] \mu, \nu[\rightarrow \mathbb{R}$ is a function, we call $Z_{\mu\nu}(h)$ the number of the zeros of h in $] \mu, \nu[$, with the following exception: if $h \equiv 0$ (i.e. h vanishes identically on the interval), $Z_{\mu\nu}(h) = -1$. In the other cases with an infinite number of roots we simply put $Z_{\mu\nu}(h) = +\infty$. We say h has a zero at μ if $\lim_{x \rightarrow \mu} h$ exists and is zero. We say h has a zero at ν if $\lim_{x \rightarrow \nu} h$ exists and is zero. We call $\bar{Z}_{\mu\nu}(h)$ the number of the zeros of h in $[\mu, \nu]$, with the same exception: if $h \equiv 0$, $\bar{Z}_{\mu\nu}(h) = -1$.

5.2. Lemma. Let $k \geq 1$ and $m \geq -1$ be two integers. Let $f :] \mu, \nu[\rightarrow \mathbb{R}$ be a k times differentiable function. If $Z_{\mu\nu}(f^{(k)}) \leq m$, then $\bar{Z}_{\mu\nu}(f) \leq k + m$. If moreover $\bar{Z}_{\mu\nu}(f) = k + m$, any root $x \in] \mu, \nu[$ is non-degenerate, i.e. $f'(x) \neq 0$.

5.3. Lemma. Consider the expression $P :]0, +\infty[\rightarrow \mathbb{R}, y \mapsto P(y) = \sum_{i=1}^n \alpha_i y^{\beta_i}$. For any $(\alpha_1, \dots, \beta_n) \in \mathbb{R}^{2n}$, $Z_{0\infty}(P) \leq n - 1$. If $n \geq 2$ and $Z_{0\infty}(P) = n - 1$, any root is non-degenerate.

Remark. This is a simplified version of Laguerre's theorem: see (1.1). To get the cases with $P \equiv 0$ we simply sum up all the monomials with same β_i and assign zero to all of the resulting coefficients. We did that in the previous proof.

Proof of Theorem 3.6. The statement to prove is $Z_{0\infty}(g) \leq 3$. We consider Expression (4.5) of $h(y)$, or rather the corresponding expression of $H(y)$ which has no denominator. Lemma 5.3 gives $Z_{0\infty}(H) \leq 3$. As $H(1) = 0$, $Z_{01}(H) \leq 2$. By Relation (4.4), $Z_{0\infty}(g'') = Z_{01}(H)$. By Lemma 5.2, $\bar{Z}_{0\infty}(g) \leq 4$. If $b > 0$, $g(0) = 0$ and $Z_{0\infty}(g) \leq 3$. In the missing case $b \leq 0$, let us suppose $Z_{0\infty}(g) = 4$.

This is the maximum number allowed, so (i) the four roots of g are non-degenerate, (ii) $b \neq 0$ (iii) $Z_{0\infty}(g'') = Z_{01}(h) = 2$, $Z_{0\infty}(h) = 3$, (iv) the three roots of h are non-degenerate, (v) the four coefficients of h are all non-zero.

By (4.5) and (v), $m_2 + m_3 \neq 0$ and $m_1 + m_3 \neq 0$. The sign of h at zero is the sign of $m_2 + m_3$. At $+\infty$ it is the sign of $-m_1 - m_3$. By (iv) this signs are opposite: $(m_2 + m_3)(m_1 + m_3) > 0$.

But $Z_{0\infty}(g) = 4$ means there are four Euler configurations such that the particle at the left has mass m_1 , the particle at the middle has mass m_2 , the particle at the right has mass m_3 . By reflection of the configuration there are also four Euler configurations such that the particle at the left has mass m_3 , the particle at the middle has mass m_2 , the particle at the right has mass m_1 . We can make the deduction above after switching the indices 1 and 3 everywhere. We get the inequality $(m_2 + m_1)(m_1 + m_3) > 0$.

By (4.2) the sign of g at zero is the sign of $m_2 + m_3$. At $+\infty$ it is the sign of $-m_1 - m_2$. According to (i) these signs coincide. This gives the conflicting inequality $(m_2 + m_3)(m_1 + m_2) \leq 0$. QED

5.4. *Non-degenerate roots.* We applied above a quite general principle: if the number of roots is maximal, the roots are non-degenerate. The principle is true in the context of Theorem 3.6: *If $\mathcal{E}_2 = 3$, the three roots of g are non-degenerate.* To prove this, we consider the only opposite case not violating the established assertion $Z_{0\infty}(g'') \leq 2$: g has a double root and two non-degenerate roots. It implies $Z_{0\infty}(g') \geq 4$ if $b > 0$, $Z_{0\infty}(g') \geq 3$ if $b \leq 0$. This is exactly the same situation as if g had four roots. It can be excluded in the same way.

5.5. *More inequalities.* Expression (4.1) is $g(s) = m_1A + m_2B + m_3C$. Assuming $b < 1$ and $s > 0$ we have $A = (1+s)((1+s)^{b-1} - 1) < 0$, $B = s(s^{b-1} - 1)$ and $C = s(1+s)(s^{b-1} - (1+s)^{b-1}) > 0$. Besides, as

$$A - B = (1+s)^{b-1} - 1 + s((1+s)^{b-1} - s^{b-1}) < 0$$

and as this expression is also $(B - C)/s$, we know that $A < B < C$. On the other hand, Expression (4.5) may be written as $h(y) = m_1\alpha + m_2\beta + m_3\gamma$ with $\alpha = 1 - y > 0$, $\beta = y^{b-2}(1 - y) > 0$, $\alpha < \beta$ and

$$\gamma = -y + \frac{2}{1-b}(1 - y^{b-1}) + y^{b-2} > 0.$$

In these inequalities we assume $0 < y < 1$ and again $b < 1$. Only the last one makes problem. To prove it, we compute $\gamma''(y) = (b-2)y^{b-3}(2y + b - 3) > 0$. So γ is convex on the interval. Together with $\gamma(1) = 0$ and $\gamma'(1) = b - 1 < 0$, this gives $\gamma > 0$.

Proof of Lemma 3.7. The first claim is satisfied if moreover $m_2 = 0$: $A < 0 < C$ implies that $g = m_1A + m_3C$ has no root. Therefore, we can prove this claim assuming $m_2 \neq 0$. Changing if necessary (m_1, m_2, m_3) into $(-m_1, -m_2, -m_3)$ we may assume $m_2 > 0$. Switching if necessary the numbering of the exterior particles 1 and 3 we may assume $m_3 \geq 0$ and $m_1 \leq 0$. If $m_1 \leq -m_2 < 0 \leq m_3$ then $g = (m_1 + m_2)A - m_2(A - B) + m_3C > 0$, the second term being positive and the other two terms non-negative according to the inequalities 5.5. Thus $Z_{0\infty}(g) = 0$. The other case is $-m_2 < m_1 \leq 0 \leq m_3$. Again by 5.5, we have $h = (m_1 + m_2)\alpha - m_2(\alpha - \beta) + m_3\gamma > 0$, which by Relation (4.4) gives $Z_{0\infty}(g'') \leq 0$. By Lemma 5.2, $Z_{0\infty}(g) \leq 2$ and if $Z_{0\infty}(g) = 2$ both roots are non-degenerate: g has the same sign at zero and at infinity. But g has the sign of $m_2 + m_3 > 0$ at zero and the sign of $-m_1 - m_2 < 0$ at infinity. So $Z_{0\infty}(g) = 1$.

The first claim is thus proved, and we know that if $\mathcal{E}_2 > 1$ necessarily $m_1m_3 > 0$. Under the conditions of the second claim, this means $0 < m_1 \leq m_3$, forgetting the equivalent $0 < m_3 \leq m_1$. Since $m_2 \geq 0$, $h = m_1\alpha + m_2\beta + m_3\gamma > 0$ by 5.5, and the signs of g at zero and $+\infty$ are distinct (implying $\mathcal{E}_2 \geq 1$). We conclude that $Z_{0\infty}(g) = 1$, exactly as we did in the case $-m_2 < m_1 \leq 0 \leq m_3$.

We now prove the last assertion of Lemma 3.7. We assume $\mathcal{E}_2 \geq 2$ in the more restrictive case $b < 0$. The previous manipulations on the masses and the

proved claims allow us to assume $m_2 < 0 < m_1 \leq m_3$. In order to conclude that $m_1 < -m_2$, we assume $-m_2 \leq m_1$ and deduce that $\mathcal{E}_2 = 1$.

First we claim that

$$g(s) = m_1(A - B + C) + (m_2 + m_1)B + (m_3 - m_1)C > 0 \quad \text{if } s \in]0, 1[. \quad (5.1)$$

As $B > 0$ if $0 < s < 1$ and $C > 0$ the last two terms are non-negative. We must simply prove that $A - B + C = 2(1+s)^b + s^{b+1} - (1+s)^{b+1} - 1 > 0$. This term is the expression of g for $(m_1, m_2, m_3) = (1, -1, 1)$. If $s = s_0$ is a root of this expression, $s = 1/s_0$ is also a root. As there are at most 3 roots, one being $s = 1$, there is at most one root, which is single, in the interval $]0, 1[$. At zero s^{b+1} is the leading term of $A - B + C$. At $s = 1$, the leading term is $(1-s)\Gamma(b)$, with $\Gamma(b) = 2^b - b - 1$. As $\Gamma''(b) > 0$, $\Gamma(0) = \Gamma(1) = 0$, we have $\Gamma(b) > 0$ if $b < 0$. Thus $A - B + C$ is positive at both extremities and consequently in all the interval $]0, 1[$. This proves (5.1). The roots of g are in $[1, +\infty[$.

We conclude by proving that $g''(s) > 0$ if $s \in]1, +\infty[$. As the dominating terms of g at $+\infty$ are $-m_1 - (m_1 + m_2)s < 0$, this means $\mathcal{E}_2 = 1$. To prove $g'' > 0$ we pass to the variable y using (4.4) and prove that $h(y)$ is positive if $y \in]1/2, 1[$. We have

$$h = m_1(\alpha - \beta + \gamma) + (m_2 + m_1)\beta + (m_3 - m_1)\gamma.$$

The last two terms are obviously non-negative and we now prove $\alpha - \beta + \gamma > 0$. For this we first note that $\alpha - \beta + \gamma = 1 - 2y + y^{b-1} + 2(1-b)^{-1}(1-y^{b-1})$ is zero at $y = 1$ and positive when $y \rightarrow 1^-$. At $y = 1/2$ it is $2^{1-b}(1-b)^{-1}\Gamma(b) > 0$, using the notation $\Gamma(b) = 2^b - b - 1$ as above. But $\alpha - \beta + \gamma$ is a Laguerre trinomial vanishing at $y = 1$, it cannot vanish more than once or have non-single root in $]1/2, 1[$, so $\alpha - \beta + \gamma > 0$ on this interval. QED

5.6. *Table.* In the previous proofs, we used the information contained in the following table

$$\left\{ \begin{array}{ll} & s \rightarrow 0 \\ \begin{array}{l} b < 1 \\ 0 < b < 1 \\ b < 0 \end{array} & \begin{array}{l} g(s) = (m_2 + m_3)s^b + \varepsilon_0 \\ \varepsilon_0 = ((b-1)m_1 - m_2 - m_3)s + o(s^b) \\ \varepsilon_0 = m_3s^{b+1} + o(s^{b+1}) \end{array} \\ & s \rightarrow +\infty \\ \begin{array}{l} b < 1 \\ 0 < b < 1 \\ b < 0 \end{array} & \begin{array}{l} g(s) = -(m_1 + m_2)s + \varepsilon_\infty \\ \varepsilon_\infty = -((b-1)m_3 - m_2 - m_1)s^b + o(s^b) \\ \varepsilon_\infty = -m_1 + o(1) \end{array} \end{array} \right. \quad (5.2)$$

In the next proof we will need the corresponding information for other orderings of the particles. While the information can be obtained by permutation of the indices of the masses, we find the following description makes it easier to grasp the basic points. As is obvious, the table describes, for a given ordering, the behavior of g when two particles tend to coincide. We use subscripts I and E (interior and exterior) to distinguish between the two particles. The sign of g at the limit is the sign of $m_I + m_E$ or $-(m_I + m_E)$ according to the collision

happens at the right ($s \rightarrow 0$, $E = 3$ and $I = 2$ in the table) or left hand side ($s \rightarrow \infty$, $E = 1$ and $I = 2$ in the table). If $b < 0$ and $m_I + m_E = 0$, g has at the limit the sign of m_E or $-m_E$, again according to the collision happens at the right or left hand side.

Proof of Theorem 3.8. To get $\mathcal{E} > 3$ we need a cell with more than one root, thus, by Lemma 3.7, two masses have the same sign and the third has a different sign, e.g. $m_1 > 0$, $m_2 < 0$, $m_3 > 0$. Cell 2 has at most three roots by Theorem 3.6. Cells 1 and 3 have at most one each by Lemma 3.7. This gives $\mathcal{E} \leq 5$.

If moreover $b < 0$, we choose the indices of the masses to get $0 < m_1 \leq m_3$. Lemma 3.7 not only gives $\mathcal{E}_1 \leq 1$, but its proof excludes the possibility of a degenerate root. Consequently $\mathcal{E}_1 = 0$ if $(m_1 + m_3)(m_1 + m_2) < 0$, and $\mathcal{E}_1 = 1$ if $(m_1 + m_3)(m_1 + m_2) > 0$. In the same way, $\mathcal{E}_3 = 0$ if $(m_3 + m_1)(m_3 + m_2) < 0$, and $\mathcal{E}_3 = 1$ if $(m_3 + m_1)(m_2 + m_3) > 0$. Our hypothesis $\mathcal{E} > 3$ would imply $\mathcal{E}_2 > 1$. By the last statement of Lemma 3.7, the allowed cases are $0 < m_1 \leq m_3 < -m_2$, $0 < m_1 < -m_2 < m_3$ and $0 < m_1 < -m_2 = m_3$. In all the cases $\mathcal{E}_1 = 0$. In the first case $\mathcal{E}_3 = 0$ and $\mathcal{E} = \mathcal{E}_2 \leq 3$. In the second case $\mathcal{E}_3 = 1$. In Cell 2, g has the same sign at infinity and at zero. Together with 5.4, this forbids $\mathcal{E}_2 = 3$. So $\mathcal{E}_2 \leq 2$ and $\mathcal{E} \leq 3$. In the last case we use 5.6. As $m_2 + m_3 = 0$, we replace $m_2 + m_3$ by the mass of the exterior colliding particle: as $(m_3 + m_1)m_2 < 0$ we get $\mathcal{E}_3 = 0$. Again $\mathcal{E} \leq 3$. QED

Proof of Theorem 3.9. See the second statement in Lemma 3.7. QED

Proof of Proposition 3.10. As seen in the proof of 3.5, when $b = 0$, $g(s) = m_2 + m_3 - (m_1 + m_2)s$. A positive root of g gives a central configuration in Cell 2. For Cell 1 and Cell 3, the corresponding equations are $m_1 + m_3 - (m_2 + m_1)s = 0$ and $m_3 + m_2 - (m_1 + m_3)s = 0$, respectively. The required conclusions follow immediately.

5.7. *Table.* We will pass to an unrestricted b , and in exchange we will only consider the case of two equal masses. When we vary the parameters (m_1, m_2, m_3, b) in \mathbb{R}^4 , the integer \mathcal{E}_2 changes only if g changes sign at $s = 0$ or at $s = +\infty$ or if it appears a degenerate root. To determine the first type of change, we give the expansions corresponding to Table (5.2) in the case $b > 1$.

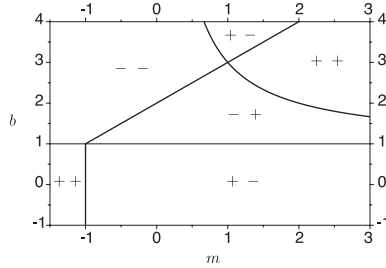
$$\left\{ \begin{array}{ll} s \rightarrow 0 & \\ 1 < b & g(s) = ((b-1)m_1 - m_2 - m_3)s + \varepsilon_0 \\ 1 < b < 2 & \varepsilon_0 = (m_2 + m_3)s^b + o(s^b) \\ 2 < b & \varepsilon_0 = b(m_1(b-1) - 2m_3)s^2/2 + o(s^2) \\ s \rightarrow +\infty & \\ 1 < b & g(s) = -((b-1)m_3 - m_2 - m_1)s^b + \varepsilon_\infty \\ 1 < b < 2 & \varepsilon_\infty = -(m_2 + m_1)s + o(s) \\ 2 < b & \varepsilon_\infty = -b(m_3(b-1) - 2m_1)s^{b-1}/2 + o(s^{b-1}) \end{array} \right. \quad (5.3)$$

Proof of Proposition 3.11. We set $(m_1, m_2, m_3) = (1, m, 1)$, where $m \in \mathbb{R}$. As the exterior masses are equal, $g(s) = 0$ if and only if $g(1/s) = 0$. We have $g(1) = 0$, which means that the symmetric configuration is always a central configuration. By Theorem 3.6, if $g \not\equiv 0$, there is at most one root $s \in]0, 1[$,

which shall be non-degenerate by 5.4. So the determination of \mathcal{E}_2 is a trivial discussion of the sign of g when $s \rightarrow 0$ and when $s \rightarrow 1^-$. The second sign is opposite to that of $g'(1) = 2b - 2^b + m(b-1)$. And when $g'(1) = 0$, which happens on the curve in Figure 3.12, 1 is a degenerate root. For the values of b satisfying respectively $b < 1$ and $1 < b$, the tables give g at $s = 0$: $(1+m)s^b$ and $(b-2-m)s$. When (m, b) is on the two half-lines, this lowest order term is zero, and the next term gives the sign of g at zero in the respective intervals $b < 0$, $0 < b < 1$, $1 < b < 2$ and $2 < b$: respectively s^{b+1} , $(b-1)s$, $(b-1)s^b$, $b(b-3)s^2/2$. A surprisingly simple result appears. In Figure 3.12, four special points $(-1, 0)$, $(-1, 1)$, $(0, 2)$, $(1, 3)$ divide the connected two half-lines into five pieces. *On each of these five pieces of line, the sign of g at zero is the same on the piece of line and on the white domain it borders.* These being checked, it is easy to finish the discussion. QED

Proof of Proposition 3.13. Instead of setting $(m_1, m_2, m_3) = (1, m, 1)$ and studying the first cell, it is equivalent to study the second cell setting $(m_1, m_2, m_3) = (m, 1, 1)$. This allows to use the tables. For the values of b satisfying respectively $b < 1$ and $1 < b$, the tables give g at $s = 0$: respectively $2s^b$ and $((b-1)m-2)s$.

For the values of b satisfying respectively $b < 1$ and $1 < b$, the tables give g at $s = +\infty$: respectively $-(1+m)s$ and $-(b-2-m)s^b$. When (m, b) is on the two half-lines of Figure 3.14, this leading term is zero. The next term, respectively 1, $(1-b)s^b$, $(1-b)s$, $b(b-3)s^{b-1}/2$, gives the sign of g at infinity in the respective intervals $b < 0$, $0 < b < 1$, $1 < b < 2$ and $2 < b$. A simple rule can be observed: on both half-lines the sign of g at infinity is the sign of g at zero.



5.8. Figure. Sign of g as $s \rightarrow 0$ and as $s \rightarrow +\infty$ for masses $m, 1, 1$.

What we need to prove is simply that, while Theorem 3.6 would allow more, g actually has the least possible number of roots allowed by the signs in the figure. To treat the part $b < 1$, it is sufficient to apply Lemma 3.7: the middle and the right hand side particles have masses with the same sign, so there is at most one root.

To treat the case $b > 1$, we start with a computation which allows to improve the general bound $\mathcal{E}_2 \leq 3$ in the case $m_2 = m_3 = 1$. Here are the expressions (4.2) and (4.5) and the second derivative of h in this case:

$$g(s) = 2s^b + (1+m)(1+s)^b + s^{b+1} - (1+s)^{b+1} - m(1+s) - s,$$

$$h(y) = -\frac{b-3}{b-1}y^{b-1} + 2y^{b-2} - (1+m)y + m - \frac{2}{b-1},$$

$$h''(y) = (b-2)(b-3)y^{b-4}(2-y).$$

As $Z_{01}(h'') \leq 0$, by Lemma 5.2 $\bar{Z}_{01}(h) \leq 2$. As $h(1) = 0$, $Z_{01}(h) \leq 1$, and $Z_{0\infty}(g'') \leq 1$ according to (4.4). Lemma 5.2 gives $\bar{Z}_{0\infty}(g) \leq 3$. In the considered case $b > 1$, $g(0) = 0$, and so, $Z_{0\infty}(g) \leq 2$. Except in the case $(m, b) = (1, 3)$ which has an infinity of roots, $\mathcal{E}_2 \leq 2$. If $\mathcal{E}_2 = 2$, the roots are non-degenerate.

Let us draw horizontal lines on Figure 5.8. We fix $b > 1$ and vary m . There is a “central interval”, empty for $b = 3$, which contains values of m such that g has different signs at zero and at infinity: on this interval $\mathcal{E}_2 = 1$. We focus on both boundary points of this central interval.

At these points $m = b - 2$ or $m = 2(b - 1)^{-1}$. In the former case, $h'(1) = b - 2 - m = 0$, thus $Z_{01}(h') \leq 0$ due to $Z_{01}(h'') \leq 0$ and to Lemma 5.2. We continue with the same arguments and get $\bar{Z}_{01}(h) \leq 1$ and $Z_{01}(h) \leq 0$ because $h(1) = 0$. In the latter case, if $b > 2$, $h(0) = m - 2(b - 1)^{-1} = 0 = h(1)$. The bound $\bar{Z}_{01}(h) \leq 2$ gives $Z_{01}(h) \leq 0$. If $1 < b \leq 2$, h is positive as $y \rightarrow 0$, and as $y \rightarrow 1^-$ because $h'(1) < 0$. So h does not vanish in $]0, 1[$, since otherwise it would be at a degenerate root or more than once. Thus $Z_{01}(h) \leq 0$ in all the cases.

Consequently for both boundary points $Z_{0\infty}(g'') \leq 0$ and $Z_{0\infty}(g) \leq 1$ since $g(0) = 0$. If $Z_{0\infty}(g) = 1$, the root is non-degenerate. As we noticed, on the half-line $m = b - 2$ the signs of g at zero and at infinity coincide: $Z_{0\infty}(g) \leq 0$. For the other point $m = 2(b - 1)^{-1}$ we have $g'(0) = (b - 1)m - 2 = 0$. So $Z_{0\infty}(g') \leq 0$. As $g(0) = 0$, $Z_{0\infty}(g) \leq 0$.

So for any $b > 1$, $Z_{0\infty}(g) = 1$ on the central interval in m and $Z_{0\infty}(g) \leq 0$ on both extremities of this interval. We observe that the factor of m in the expression of g is $(1 + s)((1 + s)^{b-1} - 1) > 0$. Consequently for values of m outside the central interval g is sign definite. QED

6. Quasi-polynomial systems, straight and curved

6.1. *Quasi-polynomial systems.* Let us write a system in two variables $x > 0$, $y > 0$:

$$0 = \sum_{i=1}^m a_i x^{b_i} y^{c_i}, \quad 0 = \sum_{i=1}^n \alpha_i x^{\beta_i} y^{\gamma_i}. \quad (6.1)$$

The system is polynomial if the exponents $b_i, c_i, \beta_i, \gamma_i$ are non-negative integers. However, as in the univariate case (1.1), we consider that $b_i, c_i, \beta_i, \gamma_i$ may be arbitrary real numbers. This extension fits well with the restriction $x > 0$, $y > 0$. What is the correct name for this class of systems? Some authors call them “fewnomial systems” (see [Kho]). But the word “fewnomial” suggests that the exponents are integers, and that there are few monomials. Other authors use it with this more restrictive meaning. Then the word does not indicate that we can vary the exponents continuously.

Before answering the question of terminology, we should notice that $(x^a - 1)/a$ belongs to the class (1.1). Its limit at $a = 0$ is $\log x$. This suggests to extend the class and accept the log. In [Ron] the quasi-polynomials are defined as expressions such as $\sum_{i=1}^k a_i x^{b_i} y^{c_i}$, but after the substitution $x = e^u$, $y = e^v$. So the quasi-polynomial functions of (u, v) have the form $\sum a_i e^{b_i u + c_i v}$. The coefficients a_i may be real numbers as in our favorite cases, but they can also be polynomials in (u, v) . A quasi-polynomial in (u, v) is a polynomial if $k = 1$, $b_1 = c_1 = 0$. The function $\log x = u$ is accepted. In [Kho] the definition of a quasi-polynomial is presented differently. One chooses a collection of k vectors $(b_i, c_i) \in \mathbb{R}^2$ and set $E_i = e^{b_i u + c_i v}$. Any polynomial in (u, v, E_1, \dots, E_k) is a quasi-polynomial. Of course, both definitions are equivalent. What may differ is the number k of basic exponential expressions E_i .

Equation (4.2) for Euler configurations of Cell 2 (ordering $x_1 < x_2 < x_3$) has the type (6.1). It is the system of two equations in $s > 0$, $t > 0$,

$$0 = s + 1 - t,$$

$$0 = (m_2 + m_3)s^b + (m_1 + m_3)t^b + m_3(s^{b+1} - t^{b+1}) - m_1 t - m_2 s.$$

Theorem 3.6 gives the optimal upper bound for the number of solutions of this system, assuming that this number is finite, while Proposition 3.5 describes the cases where this number is not finite.

Khovanskii obtained general upper bounds for the number of isolated roots of a quasi-polynomial system. It is well-known that these bounds are far from optimal, especially when applied to a particular system as ours. For a system of two equations $P_1 = P_2 = 0$ in two unknowns (u, v) , let k be the number of basic exponential expressions $E_i = e^{b_i u + c_i v}$. Let d_i , $i = 1, 2$, be the degree of the polynomial P_i in its variables (u, v, E_1, \dots, E_k) . Khovanskii's upper bound is $d_1 d_2 (d_1 + d_2 + 1)^k 2^{k(k-1)/2}$. We can play to apply this bound to our system. The first point of view is to consider that there are $k = 6$ exponential expressions $s = e^u$, $t = e^v$, e^{bu} , e^{bv} , $e^{(b+1)u}$, $e^{(b+1)v}$, and that the degrees are $d_1 = d_2 = 1$. The bound is $3^6 2^{15}$. A more successful point of view is to consider there are only $k = 4$ exponential expressions e^u , e^v , e^{bu} , e^{bv} , and that the degrees are $d_1 = 1$ and $d_2 = 2$. The bound is $2^{15} = 32768$.

Khovanskii's upper bound is obtained considering the polynomial P_2 as a function defined on the curve $P_1 = 0$. "Curved" versions of Rolle's theorem are then used to bound the number of roots. This process may be adapted to each particular problem, in order to improve the upper bound. In a recent work, [GNS] obtained in this way the following result: there are at most 12 equilibrium points in the Coulombian field of three fixed charges, except if they are infinitely many. This bound is excellent, but the authors conjecture that the optimal upper bound is 4 instead of 12.

6.2. Straight case. In our case, it is not necessary to create a curved version of Rolle's theorem. The first quasi-polynomial is $t = 1 + s$. It defines a straight segment in the quadrant $s > 0$, $t > 0$. The second quasi-polynomial is a function

on this segment, and we can apply the standard Rolle theorem after the explicit substitution $t = 1 + s$.

Leandro [Lea] proposes another problem of central configurations which can be analyzed through the standard Rolle theorem. The question is the determination of the positions of a test particle without mass under the attraction of other particles with equal masses, in such a way that there is relative equilibrium.

When we can reduce the problem to the study of a function on an interval, we say we are in the “straight case”. A system of two quasi-polynomial equations is reducible to the straight case if one of the equations is a trinomial equation $a_1 e^{b_1 u + c_1 v} + a_2 e^{b_2 u + c_2 v} = a_3 e^{b_3 u + c_3 v}$ with $(a_1, \dots, c_3) \in \mathbb{R}^9$. As we may always divide the trinomial equation by its right member, we may assume $a_3 = 1$, $b_3 = c_3 = 0$. We may set $u' = b_1 u + c_1 v$, $v' = b_2 u + c_2 v$, $x = e^{u'}$, $y = e^{v'}$. The trinomial becomes $a_1 x + a_2 y = 1$ and defines a piece of line.

A simple process of repeated differentiation with suppression of some exponential expression at each step leads to the upper bound $2^n - 2$ for the number of roots of a quasi-polynomial systems of two equations, one being a trinomial and the other a n -nomial. This easy bound is published in [LRW], where is also solved the much harder problem of the optimal upper bound in the case $n = 3$ of two trinomials. This bound is 5. It is known to be optimal thanks to a surprising example by Haas of a system of two trinomials with 5 roots. It appeared to be difficult to find examples with more than 4 roots.

The general upper bound $2^n - 2$ is 62 in our problem. It is still far above our optimal upper bound 3. We wish to emphasize that the main difficulty in finding good upper bounds, in the straight as in the curved cases, is the research of good variables, good equations, good repeated derivations, rather than the conceptuality of the tools. The lack of systematic method is quite a disappointing observation. Future progresses in this domain could come from an analysis of the successful choices. We hope that some conclusion may arrive simplifying or improving our work, [Lea], [LRW] or [GNS].

In the curved case, we do not know any work about the practical determination of the cases where the number of solutions is infinite. Probably quite subtle theorems could be used. Many such theorems are given in [Ron].

7. Open questions on central configurations

We state a problem of central configurations giving first the number n of particles and the dimension p of the configuration, which satisfies $1 \leq p \leq n - 1$. After these integer parameters, we fix the masses $(m_1, \dots, m_n) \in \mathbb{R}^n$. Finally, we decide the law of attraction: $b = -2$ and $b = -1$ correspond to configurations of vortices and celestial bodies, respectively. We can also work with any real b . We could even consider non-homogeneous attraction laws, but this would lead us too far away from our preferred applications $b = -2$ and $b = -1$.

7.1. Particles on a line. What is the maximal number of central configurations with $p = 1$ and $n = 4$, for a given $b \in \mathbb{R}$ and masses varying in \mathbb{R}^4 ? Of course, we should find together the answer to the question: when is this number

infinite? We do not know the answer, even in the case $b = -1$ of the vortices.

The known result is Moulton's theorem, which applies to the case of positive masses and $b < 0$. *There is exactly one central configuration for each ordering of the masses.* Moulton's theorem is true for any $n \geq 2$, and gives $n!/2$ central configurations. Applying this result to the case of $n = 3$ particles, we recover one of the results of Theorem 3.8: $\mathcal{E} = 3$ for $b < 0$ and positive masses. But we proved this result for any $b < 1$. Is Moulton's theorem still true if $0 \leq b < 1$?

7.2. Equal masses. The case $m_1 = m_3$ studied in Proposition 3.11 illustrates a general rule: if the repartition of the mass allows the symmetry of the configurations, and if there are few solutions, then all the solutions are symmetric. In the case $n = 4$, $p = 2$, $b = -1$ or $b = -2$, and equal masses, it appeared to be possible to prove directly the symmetry (see [Alb]). In contrast, it seems still impossible, in the case $b = -2$, to get directly a sufficiently low higher bound on the number of solutions, that would give the symmetry as a consequence.

Consider more generally a central configuration of particles with equal masses. Does it always possess some symmetry? Numerical experiments by Moeckel with $b = -2$, $p = 2$ showed that the answer is no if $n = 8$, but is probably yes if $n \leq 7$. Numerical experiments by Kathryn Glass gave the same result for $b = -1$ (see [G11], [ANS]). Glass later studied the bifurcations of the central configurations when b varies, and found asymmetric configurations with $b = -11$, $p = 2$ and $n = 6$. Faugère proved by Gröbner base techniques that for $n \leq 7$, $b = -1$, $p = 2$, all the configurations possess some symmetry. He noticed that each configuration of n indistinguishable particles $(z_1, \dots, z_n) \in \mathbb{C}^n$ is associated in a unique way with the polynomial $P(X) = (X - z_1) \cdots (X - z_n)$. He used the coefficients of P as variables and found all the central configurations for $n \leq 7$.

7.3. Four bodies in the plane. There is an outstanding recent result for $n = 4$, $p = 2$, $b = -2$. Hampton and Moeckel proved (see [HaM]) that there is a finite number of planar central configurations of 4 particles, whatever be the positive masses. This solves the first case of an old conjecture by Chazy, repeated by Wintner and then by Smale in his list of problems for the 21st century. The finiteness is also conjectured for any $n \geq 5$ and for $p = 2$ or 3. Indeed all the dimensions p between 2 and $n - 2$ are interesting and we do not know if there is finiteness when $n \geq 5$, whatever be the given set of n positive masses. In contrast, [Rob] shows that there is a continuum of central configurations if $n = 5$, $p = 2$, $b \in \mathbb{R}$, $m_1 = m_2 = m_3 = m_4 = -2^{-b}m_5$.

Hampton and Moeckel proved that there are at most 4230 central configurations with $n = 4$, $p = 2$. They used Bernstein's ideas. Between the pair "complex unknowns, positive integer exponents" and the pair "positive unknowns, real exponents", there is the interesting possibility "non-zero complex unknowns, integer exponents". Bernstein gave in 1975 a method to count the solutions with non-zero values of unknowns, and sketched a method to discuss the finiteness. The 4230 configurations are given by the "mixed volume" method, also used in [GNS]. They exist as complex solutions. Here we count them identifying a planar configuration with the reflected configuration.

We know (see [HaM]) that there are at least 10 central configurations with $n = 4$ and $p = 2$, and even 11 if we avoid an explicit degenerate case. Numerical experiments indicate 19 as the maximum. There are 19 central configurations if $m_1 = m_2 = m_3 = m_4 > 0$.

To finish, let us select an unsolved sub-problem: in the case $n = 4$, $p = 2$, positive masses, prove that there is exactly one convex central configuration such that two given particles are not adjacent (i.e. they are on the same diagonal). The existence of such a convex central configuration is known. “Convex” means that no particle is inside the triangle of the three others.

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